

PROOFS BY INDUCTION

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INTRODUCTION

This is a collection of various proofs using induction. I have tried to include many of the classical problems, such as the *Tower of Hanoi*, the *Art gallery problem*, *Fibonacci* problems, as well as other traditional examples. The problems are organized by mathematical field.

Tips for the reader: When you are reading the proofs—or better, trying to write one yourself—identify the following parts of the proof: *The base case*, *the induction hypothesis*, *where the hypothesis is used* and *where properties given to you are used*. For example, if you prove things about Fibonacci numbers, it is almost a guarantee that you have to use the recursion $f_n = f_{n-1} + f_{n-2}$ somewhere.

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1. ABSTRACT DESCRIPTION OF INDUCTION

The simplest application of proof by induction is to prove that a statement $P(n)$ is true for all $n = 1, 2, 3, \dots$. For example, “*The number $n^3 - n$ is divisible by 6*” “*The number a_n is equal to $f(n)$* ” and “*There are $n!$ permutations of n elements*” are such statements. Induction usually amounts to proving that $P(1)$ is true, and then that the implication $P(n) \implies P(n+1)$. The principle of mathematical induction can formally be stated as

$$P(1) \quad \text{and} \quad P(n) \implies P(n+1) \text{ for all } n \geq 1$$

implies that $P(n)$ is true for all $n \geq 1$.

Strong induction is similar, but where we instead prove the implication

$$(1) \quad P(1) \wedge P(2) \wedge \dots \wedge P(n) \implies P(n+1).$$

However, if we define introduce the new statement $Q(n)$ as

$$Q(n) := P(1) \wedge P(2) \wedge \dots \wedge P(n),$$

then (1) is equivalent to $Q(n) \implies Q(n+1)$. Thus, strong induction is not really different from usual induction, other than that the inductive hypothesis has a particular form.

Some textbooks introducing induction highlights the statement $P(n)$ explicitly. This is a pedagogical tool which is used to make the structure clearer. However, proofs by induction “in the wild” do not explicitly use the notation $P(n)$, the statement is simply written out. In my experience, it is easy to confuse the *statement* $P(n)$ with the *formula* one is asked to prove. For example,

“*The equality $a_n = f(n)$ holds.*”

is the statement $P(n)$ and $f(n)$ is the actual formula.

2. SUMMATION AND PRODUCTS

Problem. 1

Show that for $n \geq 1$, we have

$$(a) \sum_{k=1}^n k = \frac{n(n+1)}{2} \quad (b) \sum_{k=1}^n (2k-1) = n^2.$$

Solution. 1

(a): We first check the base case, $n = 1$. The sum is 1 and the formula evaluates to 1 as well, so we are good.

Induction hypothesis: Assume that for some $n \geq 1$ we have

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

We add $(n+1)$ on both sides of this relation and get

$$\sum_{k=1}^{n+1} k = (n+1) + \frac{n(n+1)}{2}.$$

The right hand side can now be rewritten as

$$\sum_{k=1}^{n+1} k = \frac{[n+1]([n+1]+1)}{2}.$$

Thus, we have proved that if the formula holds for n , it holds for $n+1$. By the principle of mathematical induction, the identity is true for all integers $n \geq 1$.

(b): We first check the base case, $n = 1$. Both sides evaluates to 1, so we are ok.

Induction hypothesis: Assume that for some $n \geq 1$ we have

$$\sum_{k=1}^n (2k-1) = n^2.$$

We add $2(n+1) - 1$ on both sides of this relation and get

$$\sum_{k=1}^{n+1} (2k-1) = 2(n+1) - 1 + n^2.$$

The right hand side can now be rewritten as

$$\sum_{k=1}^{n+1} (2k-1) = [n+1]^2.$$

Thus, we have proved that if the formula holds for n , it holds for $n+1$. By the principle of mathematical induction, the identity is true for all integers $n \geq 1$.

Problem. 2

Show that for $n \geq 1$, we have

$$(a) \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \quad (b) \sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}$$

Solution. 2

(a): We first check the base case, $n = 1$. Both sides evaluates to 1, so we are ok.

Induction hypothesis: Assume that for some $n \geq 1$ we have

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We add $(n+1)^2$ on both sides of this relation and get

$$\sum_{k=1}^{n+1} k^2 = (n+1)^2 + \frac{n(n+1)(2n+1)}{6}.$$

The right hand side can now be rewritten as

$$\sum_{k=1}^{n+1} k^2 = \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6}.$$

Thus, we have proved that if the formula holds for n , it holds for $n+1$. Hence, the identity is true for all integers $n \geq 1$.

(b): We first check the base case, $n = 1$. Both sides evaluates to 2, so this is done.

Induction hypothesis: Assume that for some $n \geq 1$ we have

$$\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

We add $(n+1)(n+2)$ on both sides of this relation and get

$$\sum_{k=1}^{n+1} k(k+1) = (n+1)(n+2) + \frac{n(n+1)(n+2)}{3}.$$

The right hand side can now be rewritten as

$$\sum_{k=1}^{n+1} k(k+1) = \frac{(n+1)(n+2)(n+3)}{3}.$$

Thus, we have proved that if the formula holds for n , it holds for $n+1$. Hence, the identity is true for all integers $n \geq 1$.

Problem. 3

Prove that

$$\begin{aligned} \text{(a)} \quad \sum_{j=1}^n \frac{1}{j(j+1)} &= \frac{n}{n+1} & \text{(b)} \quad \sum_{k=1}^n (-1)^k k^2 &= \frac{(-1)^n n(n+1)}{2} \\ \text{(c)} \quad \sum_{k=1}^n k \cdot k! &= (n+1)! - 1. \end{aligned}$$

Solution. 3

(a): Base case, $n = 1$, where both sides evaluates to $\frac{1}{2}$. *Induction hypothesis:* Assume that for some $n \geq 1$ we have

$$\sum_{j=1}^n \frac{1}{j(j+1)} = \frac{n}{n+1}.$$

We add $1/((n+1)(n+2))$ on both sides of this relation and get

$$\sum_{j=1}^{n+1} \frac{1}{j(j+1)} = \frac{1}{(n+1)(n+2)} + \frac{n}{n+1}.$$

Some algebra in the right hand side gives

$$\sum_{j=1}^{n+1} \frac{1}{j(j+1)} = \frac{n+1}{n+2}.$$

Thus, we have proved that if the formula holds for n , it holds for $n+1$, and induction then tells us that the formula is true for all $n \geq 1$.

(b): Base case, $n = 1$, where both sides evaluates to -1 . *Induction hypothesis:* Assume that for some $n \geq 1$ we have

$$\sum_{k=1}^n (-1)^k k^2 = \frac{(-1)^n n(n+1)}{2}.$$

We add $(-1)^{n+1}(n+1)^2$ on both sides:

$$\sum_{k=1}^{n+1} (-1)^k k^2 = (-1)^{n+1}(n+1)^2 + \frac{(-1)^n n(n+1)}{2}.$$

This simplifies to

$$\sum_{k=1}^{n+1} (-1)^k k^2 = \frac{(-1)^{n+1}(n+1)(n+2)}{2}.$$

Thus, we have proved that if the formula holds for n , it holds for $n+1$, and induction then tells us that the formula is true for all $n \geq 1$.

(c): The base case $n = 1$ says $1 \cdot 1! = 2! - 1$, which is true. Assume that the formula holds for a particular value of $n \geq 1$. We add $(n+1)(n+1)!$ on both sides and get

$$\sum_{k=1}^n k \cdot k! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)!$$

$$\Longleftrightarrow$$

$$\sum_{k=1}^{n+1} k \cdot k! = (n+1)![1 + (n+1)] - 1.$$

Here, we rewrote the sum and the right hand side a bit. Now we note that this is simply

$$\sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1,$$

which is the formula for $n+1$. This completes the proof.

Problem. 4

Formula for geometric sum. Suppose $r \neq 1$ and prove that

$$\sum_{j=0}^n r^j = \frac{r^{n+1} - 1}{r - 1}.$$

Solution. 4

Base case $n = 0$: The left hand side is just $r^0 = 1$. The right hand side is $\frac{r-1}{r-1} = 1$ as well. Suppose now that the formula holds for a particular value of n . We wish to prove that

$$\sum_{j=0}^{n+1} r^j = \frac{r^{n+2} - 1}{r - 1}.$$

This is equivalent to proving

$$r^{n+1} + \sum_{j=0}^n r^j = \frac{r^{n+2} - 1}{r - 1},$$

and using the induction hypothesis, the sum in the left hand side can be expressed using the formula. Thus, we need to prove

$$r^{n+1} + \frac{r^{n+1} - 1}{r - 1} = \frac{r^{n+2} - 1}{r - 1}.$$

Multiplying both sides with $r - 1$ gives

$$(r - 1)r^{n+1} + r^{n+1} - 1 = r^{n+2} - 1,$$

which is easily seen to be true. Hence, the formula above holds for all $n \geq 0$.

Problem. 5

Prove that $n! > 2^n$ for $n \geq 4$.

Solution. 5

The base case $n = 4$ states that $4! > 2^4$, and we see that $24 > 16$ so this is true. Suppose the inequality holds for a particular $n \geq 4$, so that

$$n! > 2^n.$$

Multiply both sides with $(n + 1)$, and we then know

$$(n + 1)n! > (n + 1)2^n \iff (n + 1)! > (n + 1)2^n.$$

Note that $n + 1 > 2$, so for sure $(n + 1)2^n > 2 \cdot 2^n = 2^{n+1}$. We can therefore conclude that

$$(n + 1)! > 2^{n+1}.$$

Thus, we have proved that

$$n! > 2^n \Rightarrow (n + 1)! > 2^{n+1} \text{ whenever } n \geq 4$$

and so the inequality holds for all integers $n \geq 4$.

Problem. 6

Prove that

$$(a) \prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}, \quad (b) \prod_{j=1}^n \frac{j^2}{j+1} = \frac{n!}{n+1}.$$

Solution. 6

(a): Base case is $n = 2$. The left hand side is just $1 - \frac{1}{4}$ while the right hand side is $\frac{3}{4}$, so both sides are equal.

Suppose now that

$$\prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n} \text{ for some } n \geq 2.$$

After multiplying both sides with $1 - \frac{1}{(n+1)^2}$ we get

$$\begin{aligned} \prod_{j=2}^{n+1} \left(1 - \frac{1}{j^2}\right) &= \left(1 - \frac{1}{(n+1)^2}\right) \frac{n+1}{2n} \\ &= \frac{(n+1)^2 - 1}{(n+1)^2} \frac{n+1}{2n} \\ &= \frac{(n+1)^2 - 1}{2n(n+1)} \\ &= \frac{(n+1) + 1}{2(n+1)}. \end{aligned}$$

Thus, the induction hypothesis implies

$$\prod_{j=2}^{n+1} \left(1 - \frac{1}{j^2}\right) = \frac{(n+1) + 1}{2(n+1)},$$

which is the formula for “the next n ”. This concludes the proof.

(b): Same idea, we check the base case $n = 1$, and see that both sides are equal to $\frac{1}{2}$. Suppose the formula holds for some $n \geq 1$, and multiply both sides with $\frac{(n+1)^2}{(n+1)+1}$. The result we get is

$$\begin{aligned} \prod_{j=1}^{n+1} \frac{j^2}{j+1} &= \frac{(n+1)^2}{(n+1)+1} \frac{n!}{n+1} \\ &= \frac{n!(n+1)}{n+2} \\ &= \frac{(n+1)!}{n+2} \end{aligned}$$

which is what we want to prove.

Problem. 7

Prove that

$$\prod_{j=1}^n \left(1 + \frac{1}{j^2}\right) \leq n + 1.$$

Solution. 7

The inequality is certainly true for $n = 1$, as we get $3/2 < 2$. Suppose the inequality is true for some $n \geq 1$. Multiply both sides with $1 + (n+1)^{-2}$ and we get

$$\prod_{j=1}^{n+1} \left(1 + \frac{1}{j^2}\right) \leq (n+1)(1 + (n+1)^{-2}).$$

Thus, we conclude that

$$\prod_{j=1}^{n+1} \left(1 + \frac{1}{j^2}\right) \leq (n+1) + \frac{1}{n+1} < n+2$$

since for sure $\frac{1}{n+1} < 1$. We conclude that the induction hypothesis implies

$$\prod_{j=1}^{n+1} \left(1 + \frac{1}{j^2}\right) < n+2$$

and we are done.

Problem. 8

Prove that

$$\sum_{k=1}^n \frac{1}{k^2} < 2.$$

Hint: Prove the stronger statement that the left hand side is less than $2 - \frac{1}{n}$.

Solution. 8

This is an example where it is sometimes easier to prove a stronger result. The base case $n = 1$ is clear. *Induction hypothesis:* Let $n \geq 1$ and suppose

$$\sum_{k=1}^n \frac{1}{k^2} < 2 - \frac{1}{n}.$$

Add $1/(n+1)^2$ on both sides. We then get

$$\sum_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 - \frac{(n+1)^2 - n}{n(n+1)^2}.$$

Now note that

$$2 - \frac{(n+1)^2 - n}{n(n+1)^2} = 2 - \frac{n^2 + n + 1}{n(n+1)^2} < 2 - \frac{n(n+1)}{n(n+1)^2} = 2 - \frac{1}{n+1}$$

where the inequality comes from removing the 1 in the numerator. Thus,

$$\sum_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n+1}$$

and we are done.

Problem. 9

Try to prove that

$$\prod_{k=1}^n \frac{2k-1}{2k} < \frac{1}{\sqrt{3n}}.$$

If you fail, try to prove the stronger result that the left hand side is less than or equal to $\frac{1}{\sqrt{3n+1}}$.

Solution. 9

The base case $n = 1$ is easy to verify, we find that both sides are equal to $\frac{1}{2}$.

Suppose now that $\prod_{k=1}^{n-1} \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3(n-1)+1}}$. We then have that

$$\prod_{k=1}^n \frac{2k-1}{2k} = \left(\prod_{k=1}^{n-1} \frac{2k-1}{2k} \right) \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3(n-1)+1}} \frac{2n-1}{2n}$$

where we have used the induction hypothesis for the product in the middle expression. To finish the proof, we must show that

$$\frac{1}{\sqrt{3(n-1)+1}} \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

Clearing denominators gives the equivalent inequality

$$(2n-1)\sqrt{3n+1} \leq 2n\sqrt{3n-2}.$$

We can now square both sides (as squaring of non-negative numbers preserve inequalities)

$$(2n-1)^2(3n+1) \leq (2n)^2(3n-2).$$

Hence, it is enough to prove that

$$0 \leq (2n)^2(3n-2) - (2n-1)^2(3n+1).$$

But the right hand side simplifies to $n-1$, and since $n \geq 1$, we can be sure that $n-1 \geq 0$, and we are done.

To recap,

$$\prod_{k=1}^n \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3(n-1)+1}} \frac{2n-1}{2n}$$

by using the induction hypothesis on the first $n-1$ terms in the product. Then, we proved that

$$\frac{1}{\sqrt{3(n-1)+1}} \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$$

by the argument above, so together,

$$\prod_{k=1}^n \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3n+1}}.$$

Finally, it is straightforward to see that $\frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{3n}}$, as we make the denominator smaller. Hence, we arrive at

$$\prod_{k=1}^n \frac{2k-1}{2k} < \frac{1}{\sqrt{3n}}.$$

3. CALCULUS

Problem. 10

Show that $D[x^n] = nx^{n-1}$. You are allowed to use the chain rule, and that $D[x] = 1$.

Problem. 11

Let $x > -1$ and $n \in \mathbb{N}$. Prove that $(1+x)^n \geq 1+nx$.

Problem. 12

Let $x \neq 0$ be a real number such that $x + x^{-1} \in \mathbb{Z}$. Prove that for any $n \in \mathbb{Z}$, $x^n + x^{-n} \in \mathbb{Z}$.

Hint: You need two base cases.

Solution. 12

Because of symmetry, it suffices to prove this whenever $n \geq 0$. Base cases are $n = 0$ and $n = 1$, which are easy to verify.

Induction hypothesis: we suppose that $x^{n-1} + x^{-(n-1)}$ and $x^n + x^{-n}$ are integers. Since both $x^n + x^{-n}$ and $x + x^{-1}$ are integers, their product

$$(x^n + x^{-n})(x + x^{-1}) = [x^{n+1} + x^{-(n+1)}] + [x^{n-1} + x^{-(n-1)}]$$

is an integer. However, the induction hypothesis states that the second bracket is an integer, so then the first bracket must also be an integer. That is, $x^{n+1} + x^{-(n+1)} \in \mathbb{Z}$, and we are done.

Problem. 13

Suppose $z = a(\cos \theta + i \sin \theta)$ is a complex number. Show that

$$z^n = a^n [\cos(n\theta) + i \sin(n\theta)].$$

You will need the addition formulas for the trigonometric functions.

Problem. 14

Show that for all $n \geq 0$, we have

$$\int_0^{\infty} x^n e^{-x} dx = n!.$$

Solution. 14

The base case $n = 0$ is $\int_0^{\infty} e^{-x} dx = [-e^{-x}]_0^{\infty} = 1$, which agrees with $0!$.

Suppose now that $\int_0^{\infty} x^n e^{-x} dx = n!$. We have that for $n \geq 1$, that

$$\begin{aligned} \int_0^{\infty} x^{n+1} e^{-x} dx &= \text{(Using integration by parts)} \\ &= [x^{n+1}(-e^{-x})]_0^{\infty} - \int_0^{\infty} (n+1)x^n(-e^{-x})dx \\ &= 0 + (n+1) \int_0^{\infty} x^n e^{-x} dx \\ &= 0 + (n+1)n! \end{aligned}$$

where we in the last step, we used the induction hypothesis. Since $(n+1)! = (n+1)n!$, we are done.

4. RECURSION

Problem. 15

Let $a_0 = 0$ and $a_n = 2a_{n-1} + n$ whenever $n \geq 1$. Show that $a_n = 2^{n+1} - n - 2$.

Solution. 15

We first verify the formula for $n = 0$, which is easy. We now assume that the formula holds for $n - 1$. That is, our induction hypothesis is $a_{n-1} = 2^{(n-1)+1} - (n-1) - 2$. The recursion now tell us that

$$a_n = 2 \left(2^{(n-1)+1} - (n-1) - 2 \right) + n$$

where we have substituted a_{n-1} with our hypothesis. Simplifying the expression gives

$$a_n = 2 \cdot 2^n - 2(n-1) - 4 + n = 2^{n+1} - n - 2,$$

which is exactly what we want to prove.

Problem. 16

Let $a_0 = 0$, $a_1 = 1$ and $a_{n+2} = \frac{1}{4}(a_{n+1}^2 + a_n + 2)$ whenever $n \geq 0$. Show that $0 \leq a_n \leq 1$ for all n .

Solution. 16

We have that the statement is true for $n = 0$ and $n = 1$. Assume now that $0 \leq a_n, a_{n+1} \leq 1$ for some $n \geq 0$. Using these assumptions, we see that

$$\frac{1}{4}(0^2 + 0 + 2) \leq \frac{1}{4}(a_{n+1}^2 + a_n + 2) \leq \frac{1}{4}(1^2 + 1 + 2).$$

Simplifying, we obtain

$$\frac{1}{2} \leq a_{n+2} \leq 1,$$

and we may conclude that $0 \leq a_{n+2} \leq 1$ also.

Problem. 17

Let $a_0 = 1$ and $a_n = \sum_{i=0}^{n-1} a_i$. Prove that¹ for $n \geq 1$, we have $a_n = 2^{n-1}$.

Solution. 17

Use strong induction.

Problem. 18

Let $a_1 = 1$ and $a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ whenever $n \geq 2$. Show that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \geq 1$.

Solution. 18

The formula tell us that $a_1 = 3 \cdot 2^0 + 2(-1) = 1$ and $a_2 = 3 \cdot 2^1 + 2 = 8$, which agree with the definition of a_1 and a_2 . Hence we have two base cases.

Assume now that

$$a_{n-1} = 3 \cdot 2^{(n-1)-1} + 2(-1)^{n-1} \text{ and } a_{n-2} = 3 \cdot 2^{(n-2)-1} + 2(-1)^{n-2}.$$

With this assumption together with the recursion,

$$a_n = \left(3 \cdot 2^{(n-1)-1} + 2(-1)^{n-1}\right) + 2 \left(3 \cdot 2^{(n-2)-1} + 2(-1)^{n-2}\right).$$

It is now a matter of algebra to see that the right hand side can be rewritten as $3 \cdot 2^{n-1} + 2(-1)^n$, and we are done.

Problem. 19

Let $a_0 = 0$ and $a_{n+1} = \sqrt{2a_n + 3}$ whenever $n \geq 0$. Show that $0 \leq a_n \leq 3$.

4.1. Fibonacci numbers. The Fibonacci numbers are perhaps the most famous sequence of numbers defined via a recursion. Since recursion and induction play very nice together, it is not a surprise that there are many exercises involving the Fibonacci numbers.

Problem. 20

The Fibonacci sequence is defined as $f_0 = 0$, $f_1 = 1$ and

$$f_n = f_{n-1} + f_{n-2} \text{ whenever } n \geq 2.$$

Show that

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Solution. 20

We need two base cases: $n = 0$ and $n = 1$. These are easy to verify.

Assume that the closed-form formula holds for $m-1$ and $m-2$. Let $\beta = \frac{1}{2}(1+\sqrt{5})$ and $\alpha = \frac{1}{2}(1-\sqrt{5})$. Note that β and α are the two solutions to the equation

$$(*) \quad t^2 - t - 1 = 0.$$

Using the recursion for the Fibonacci numbers, it is then enough to verify that

$$\begin{aligned} \frac{1}{\sqrt{5}} [\beta^m - \alpha^m] &= \frac{1}{\sqrt{5}} [\beta^{m-1} - \alpha^{m-1}] + \frac{1}{\sqrt{5}} [\beta^{m-2} - \alpha^{m-2}] \\ &\iff \\ \beta^m - \alpha^m &= \beta^{m-1} - \alpha^{m-1} + \beta^{m-2} - \alpha^{m-2} \\ &\iff \\ \beta^m - \beta^{m-1} - \beta^{m-2} &= \alpha^m - \alpha^{m-1} - \alpha^{m-2} \end{aligned}$$

¹Note that this formula does not hold for $n = 0$.

$$\beta^{m-2} \underbrace{(\beta^2 - \beta - 1)}_{=0} = \alpha^{m-2} \underbrace{(\alpha^2 - \alpha - 1)}_{=0}.$$

The equality with 0 on both sides follows from the fact that β and α are solutions to (*). This concludes the proof.

Problem. 21

Show that for the Fibonacci numbers,

$$f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}.$$

Problem. 22

Show that for the Fibonacci numbers,

$$(a) \sum_{i=1}^n f_i = f_{n+2} - 1 \quad (b) \sum_{i=1}^n f_i^2 = f_n f_{n+1}.$$

Problem. 23

Define the matrix A as below and show the formula for A^n :

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

where the f_i are the Fibonacci numbers.

Problem. 24

Show that for the Fibonacci numbers,

$$(a) \gcd(f_{n+1}, f_n) = 1 \quad (b) \gcd(f_m, f_n) = f_{\gcd(m,n)} \text{ for } m, n \geq 0.$$

Solution. 24

(a) We first check that $\gcd(f_1, f_2) = 1$ which is true. Then, by using the fact that $\gcd(a+b, b) = \gcd(a, b)$, we have that

$$\gcd(f_{n+1}, f_n) = \gcd(f_n + f_{n-1}, f_n) = \gcd(f_{n-1}, f_n)$$

and the last expression is 1 by induction.

(b) See <http://www.cut-the-knot.org/arithmetic/algebra/FibonacciGCD.shtml>

5. NUMBER THEORY

Problem. 25

Show that $n^3 - n$ is divisible by 6.

Problem. 26

Show that $6^n - 1$ is divisible by 5.

Problem. 27

Show that 3^{n+1} divides $2^{3^n} + 1$.

Solution. 27

The base case $n = 0$ is clear. Assume 3^{n+1} divides $2^{3^n} + 1$ for some $n \geq 0$. This means that $2^{3^n} + 1 = m3^{n+1}$ for some m (which depends on n), and $2^{3^n} = m3^{n+1} - 1$. Now, the next step is

$$\begin{aligned} 2^{3^{n+1}} + 1 &= \left(2^{3^n}\right)^3 + 1 \\ &= (m3^{n+1} - 1)^3 + 1 \\ &= (m3^{n+1})^3 - 3(m \cdot 3^{n+1})^2 + 3m \cdot 3^{n+1} - 1 + 1 \\ &= m^3 3^{3(n+1)} - 3m^2 \cdot 3^{2(n+1)} + 3m3^{n+2} \\ &= 3^{n+2} (m^3 3^{2n+1} - 3m^2 \cdot 3^n + m), \end{aligned}$$

and this is now a multiple of 3^{n+2} .

Problem. 28

Show that every integer $n \geq 1$ has a prime factorization.

Solution. 28

An integer $n \geq 2$ is either a prime number, or divisible by some k , with $1 < k < n$.

Base case: $n = 1$ is our base case. It is a product of 0 prime numbers. We use strong induction: let $n \geq 2$ and assume that every integer less than n can be expressed as a product of primes. There are two cases to consider:

- n is a prime, in which case it has a prime factorization, namely n as a single factor.
- otherwise, n is composite, so $n = k \cdot m$ for some integers k and m , with $2 \leq k, m < n$. By induction hypothesis, each of the integers k and m can be factored into primes. This means that the product $k \cdot m$ is also a product of primes, so n is a product of prime numbers. In other words, n has a prime factorization.

Problem. 29

Let p be a prime number. Show that $k^p \equiv_p k$ by induction over k .

You may use the *Freshman's dream*: $(a + b)^p \equiv_p a^p + b^p$.

Problem. 30

Show that for any $n \geq 3$, one can find a set A consisting of n different positive integers, such that the sum of the numbers is divisible by every element in A .

Solution. 30

For the base case $n = 3$, we can take $A = \{1, 2, 3\}$. Suppose now we have such a set $A_n = \{a_1, \dots, a_n\}$. Let $s = a_1 + a_2 + \dots + a_n$ so that $a_i | s$ for $i = 1, \dots, n$. Now consider the new set

$$A_{n+1} = \{a_1, a_2, \dots, a_n, s\}.$$

The sum of the entries in A_{n+1} is $2s$, which is divisible by s and all other entries. Thus we have a set with $n + 1$ elements with the sum of the entries divisible by each member.

Problem. 31

Show that for any $n \geq 3$, one can find different positive integers $a_1, a_2, \dots, a_n$ such that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

For example, for $n = 3$ we have $1 = 1/2 + 1/3 + 1/6$.

Solution. 31

The base case: $n = 3$ was given in the problem.

Suppose that

$$(2) \quad 1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

with $a_1 > a_2 > \cdots > a_n$. We then have that

$$(3) \quad 1 = \frac{1}{2} + \frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_n}$$

so if $\{a_1, \dots, a_n\}$ is a sequence of denominators that solve the problem, the sequence

$$\{2, 2a_1, 2a_2, \dots, 2a_n\}$$

is a sequence of denominators of length $n + 1$ which also solves the problem.

Alternative solution: The base case: $n = 3$ was given in the problem.

Suppose that

$$(4) \quad 1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

with $a_1 > a_2 > \cdots > a_n$. The tricky part is to notice that

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$$

where the last identity is easy to verify via algebra. Thus, if we replace $1/a_n$ in (4) with what the above formula gives, we have an expression with one more fraction: That is,

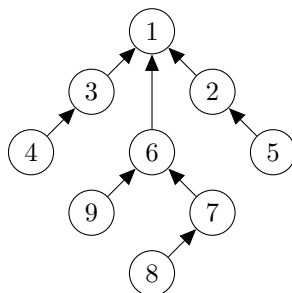
$$(5) \quad 1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n + 1} + \frac{1}{a_n(a_n + 1)}$$

and it is clear that all denominators are still different. Hence, we have a solution with $n + 1$ terms.

6. COUNTING

Problem. 32

A *decreasing tree* is a tree where the vertex labels appear in a decreasing manner on every path from a vertex to the root. Show that the number of such decreasing trees on n vertices is $(n - 1)!$. For example, the following labeled tree is such a tree:



7. LINEAR ALGEBRA AND MATRICES

Problem. 33

The square $2n \times 2n$ -matrix C_n is defined as

$$(\delta_{ij} + 2\delta_{i,2n-j+1})_{ij},$$

where we use the Kronecker delta. For example,

$$C_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find and prove a formula for $\det(C_n)$.

Solution. 33

We first compute $|C_1|$, $|C_2|$ och $|C_3|$. The corresponding values are -3 , 9 and -27 . It seems reasonable to believe that $|C_n| = (-3)^n$, so this is what we shall prove. The base case $n = 1$ is already done.

Suppose now $|C_n| = (-3)^n$ for some fixed $n \geq 1$. The goal is to show that $|C_{n+1}| = (-3)^{n+1}$. Note that we can write C_{n+1} in block matrix form,

$$C_{n+1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & C_n & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

where 0 represents a row or a column with only zeros, of appropriate dimensions. We expand the determinant of C_{n+1} along its first column, and we have

$$|C_{n+1}| = \begin{vmatrix} C_n & 0 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ C_n & 0 \end{vmatrix}.$$

Note that since there is an odd number of rows, we get a minus sign at the end. We now expand both remaining determinants along the last column. We now obtain

$$|C_{n+1}| = |C_n| - 4|C_n| = -3|C_n|.$$

Hence, $|C_{n+1}| = -3|C_n|$. Now if $|C_n| = (-3)^n$, we must have $|C_{n+1}| = (-3)^{n+1}$, and we are done.

8. CONSTRUCTIONS

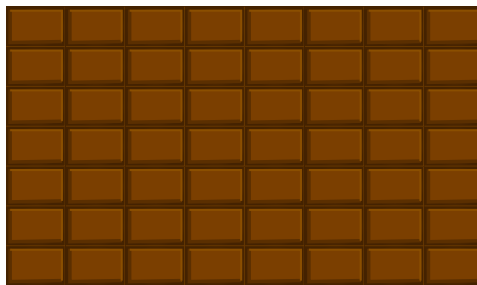
Problem. 34

Consider a standard bar of chocolate — a rectangle with n bits. You may break the bar along the ridges into two smaller pieces and repeat this procedure on the smaller pieces until you have n bits. Show that you need to do exactly $n - 1$ such breaks, no matter of the strategy.

Solution. 34

The base case is $n = 1$, in which case there are no possible breaks. We use strong induction and assume that every bar with $1 \leq k < n$ bits requires $k - 1$ breaks to be turned into bits. Consider a bar B with n bits, which we break into pieces of size a and b , with $1 \leq a, b < n$. The total number of breaks to turn B into bits is then $1 + (a - 1) + (b - 1)$ due to the induction hypothesis — one break to split B and then the breaks needed to turn the two pieces into bits.

But since $a + b = n$, we have that $1 + (a - 1) + (b - 1) = n - 1$ which proves that even bars of size n requires $n - 1$ breaks.

FIGURE 1. A standard bar of chocolate, $n = 56$.**Problem. 35**

Consider n circles in the plane. They separate the plane into regions. Show that one can color the regions red or blue, such that no two adjacent regions share the same color.

Problem. 36

The towers of Hanoi is a classical puzzle. You have three pillars, with n gold plates on the first pillar, all in different sizes with the smaller plates above the larger plates. Your task is to move one plate at the time, such that you end up with all n plates at the last pillar. During this procedure, a larger plate is never allowed to be on top of a smaller plate.

Show that this can be done using $2^n - 1$ moves.

Solution. 36

For $n = 1$, there is only one plate, and we can simply move it from the first to the third pillar, requiring 1 move.

Note that if we can move the pile from the first to the third pillar, we can also move it from the first to the second pillar in the same manner.

Thus, to move n plates, we first move the top $n - 1$ plates to the second pillar (using $2^{n-1} - 1$ moves), then move the remaining largest plate from the first to the third pillar, and then finally move the $n - 1$ plates from the second pillar to the third pillar, again using $2^{n-1} - 1$ moves.

The total number of moves with n plates is therefore $2(2^{n-1} - 1) + 1 = 2^n - 1$, as predicted by the formula.

Problem. 37

Consider a $2^n \times 2^n$ -grid, with one square removed. Show that the remaining squares can be tiled with L -triominoes. An L -triomino is a figure with the shape .

Problem. 38

Consider triangle consisting of 4^n smaller triangles. Remove one of the three corners. Show that the remaining triangles can be tiled with Δ -pieces.

Solution. 38

The base case $n = 1$ is clear — the figure consists of one tile.

Assume now we can tile the $n - 1$ case, and consider a size- n triangle. Such a triangle can be divided into four $(n - 1)$ -triangles. We can add one of the tiles such as it covers three different corners. Now, all four $(n - 1)$ -triangles are missing a corner, and we can tile each of these via the induction hypothesis.

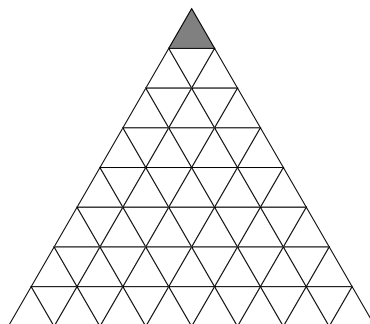
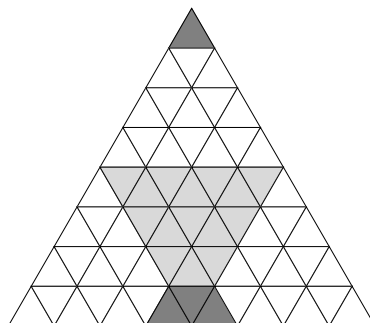
FIGURE 2. The case $n = 3$ with 64 triangles.

FIGURE 3. The triangle divided into 4 smaller triangles, and the added tile.

Problem. 39

Show that every integer $n \geq 8$ can be expressed as $n = 3k + 5m$ for some integers $k, m \geq 0$.

Solution. 39

Suppose $n > 8$. Our induction hypothesis is that we have such an expression for $n - 1$. There are two cases to consider:

- either $n - 1$ has a 5 in the representation, in which case we remove it, and add two 3's, which gives a valid representation for n , or
- $n - 1$ has only 3's in the representation. Since $n > 8$, the number of 3's is at least three, so we can remove three of them, and add two 5's instead. This gives a valid representation of n .

Problem. 40

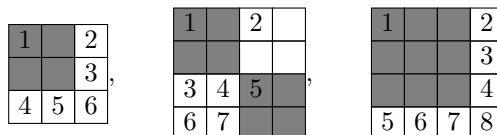
Prove that any square can be subdivided into any number of squares $n \geq 6$.

Any cube can be subdivided into any number of cubes ≥ 47 . For hypercubes, this is still an open problem, see <https://arxiv.org/pdf/1910.06206.pdf>.

Solution. 40

We need to show that for any $n \geq 6$, there is a square which is subdivided into n subsquares.

Our base cases are the following: There are squares consisting of $n = 6$, $n = 7$ and $n = 8$ subsquares:



Suppose now $n > 8$ and that we can construct a square S with $(n - 3)$ subsquares. We can then construct a square with n subsquares as follows. Take S and add three additional squares of the same size, forming a larger square:

$$S \quad \longrightarrow \quad \begin{array}{|c|c|} \hline S & 1 \\ \hline 2 & 3 \\ \hline \end{array}.$$

This proves that every number > 5 of subsquares can be obtained — from the first base case we can obtain the numbers 6, 9, 12, 15, ... The second base case gives 7, 10, 13, 16, ... and the last base case gives 8, 11, 14, 17, ... These sequences cover all integers greater than 5.

Problem. 41

There are n pirates who will split a pile of gold such that each pirate gets a fair share of the treasure. Pirates are not objective, and each pirate has its own view of what is a $\frac{1}{n}$ th part of the treasure. Devise a strategy to divide the gold among the pirates.

Solution. 41

We do induction over n . Base case is one pirate who is happy with the entire treasure.

Induction hypothesis: $n - 1$ pirates can fairly divide a treasure. We now consider n pirates. Take one piece of gold at a time from the treasure and put in a separate pile P . As soon as some pirate believes that P is a fair share, he takes gets that pile and is happy. The remaining $(n - 1)$ pirates must then believe that the remaining treasure is at least $(n - 1)/n$ of the original treasure, or they would have preferred the pile at an earlier stage. By induction, the remaining $n - 1$ pirates can now proceed to divide the remaining treasure amongst themselves.

9. GEOMETRY

Problem. 42

Show that the number of diagonals in a convex n -gon is $\frac{1}{2}n(n - 3)$.

Problem. 43

Show that n lines in the plane in *general position* (no parallel lines, no three intersect in a single point) separates the plane into $\frac{1}{2}n(n + 1) + 1$ regions.

Solution. 43

The base case $n = 1$ gives two regions, which agrees with the formula.

Now consider n lines, which we assume divides the plane into $\frac{1}{2}n(n + 1) + 1$ regions. We add an additional line L . This line intersects the n previous lines. The intersection points divides L into $n + 1$ segments, and each such segment bisects a previous region. Hence, we have $n + 1$ additional regions so $n + 1$ lines produces

$$\frac{n(n + 1)}{2} + 1 + (n + 1) = \frac{n(n + 1) + 2(n + 1)}{2} + 1 = \frac{(n + 1)(n + 2)}{2} + 1$$

regions, which is what the formula predicts.

Problem. 44

Show that any polygon in the plane can be triangulated without introducing additional vertices.

Solution. 44

We do induction over the number of vertices, n . When $n = 3$, we have a triangle so this case is ok.

Strong induction hypothesis: Assume that we can triangulate any polygon with at most $n - 1$ vertices, where $n > 3$. Note now that it is enough to find a diagonal in the polygon, connecting two vertices. This cuts the polygon into two polygons, each with at most $n - 1$ vertices. These can then be triangulated by the induction hypothesis.

To find such a diagonal, take an edge with vertices U, V of the polytope, and let M be the midpoint. Move M inwards the polytope along the line perpendicular to the edge, and consider the line segments UM and VM . Eventually, at least one of these line segments intersect some other edge E at a point M' . We then let M' slide along E such that the distance total distance $|UM'| + |VM'|$ increases².

Eventually, one of line segments UM' or VM' will intersect a vertex of the polytope (think about why this is true!). Let us call this vertex W . Then at least one of the line segments UW or VW can be chosen as the diagonal we seek.

Problem. 45

The art gallery problem. An art gallery is in the shape of a (perhaps non-convex) polygon with n vertices. Show that one requires at most $\lfloor n/3 \rfloor$ guards, placed at some of the vertices, such that they together can see the entire art gallery.

Hint: Use the fact that the art gallery can be triangulated. Show that one can color the vertices in the triangulation with three colors, such that the colors in every triangle are different.

Solution. 45

We show that such a coloring exist by induction over n . The base case $n = 3$ is clear — that is a triangle.

Strong induction hypothesis: Suppose that we can color any polygon with at most $n - 1$ vertices in the described manner. Let P be a triangulated polygon with n vertices. We can find a diagonal that cuts P into two smaller polygons, Q and R . By induction, both these can be colored such that every triangle has different colored vertices.

We need to turn the colorings of Q and R into a coloring of P . If the colors of the two vertices of the diagonal match up, we can simply glue Q and R back together and we are done. Otherwise, we can permute the colors of Q , such that the two shared vertices with R match up, and we can glue them together to obtain a coloring of P . This finishes the induction part of the proof.

Finally, since we have n vertices with three different colors, there must be some color that only appears $\lfloor n/3 \rfloor$ times, say red. We put a guard at each red vertex. Since every triangle has a red vertex, every triangle is covered by at least one guard.

10. GRAPH THEORY

²Geometrically, we have an ellipse with focal points U and V , and the point M' on the ellipse, and E being a line segment passing through M' . We let M' slide along E so that it exits the ellipse.

Problem. 46

Show that the number of edges in a tree with n vertices is $n - 1$.

Problem. 47

Euler's Theorem. Let G be a connected graph where every vertex has even degree. Then G has an Eulerian cycle, that is, a cycle visiting every edge in G .

You may use that every such graph G has *some* cycle.

Proof. We use strong induction over the number of edges. If there are 0 edges, G is a single vertex, which is considered to be an Eulerian cycle.

Suppose now we have n edges in G , and consider a cycle C of G . Remove all edges C from G — this might disconnect the graph into k components, say $H_1, H_2, \dots, H_k$. Each H_i is a connected graph where every vertex has even degree (why?) and thus has an Eulerian cycle.

We can now form a big Eulerian cycle in G as follows. We follow the edges in the cycle C , but whenever we hit a vertex in some (unvisited) H_i , we take a detour, visiting all edges in H_i along its Eulerian cycle, and then continue along C . $\square$

Problem. 48

A country is divided into n states and one can travel the country and visit all states.

The tourism in the country is unfortunately in decline, so the queen comes up with the idea that the states should be divided into regions, such that a tourist can start somewhere in the country, visit all regions exactly once, and end up in the first region.

Show that the queen's wish can be fulfilled and that at most $2n - 1$ regions are required.

Solution. 48

It is clear that if there is only one state, we only need one region and we are done by default.

Suppose now that we have a country with n states and $2n - 1$ regions. We add a new additional state N to the country, adjacent to (at least one) region R in the original country. By induction, there is an itinerary that visits all regions in the original country. We modify this itinerary, by dividing R into two smaller regions, R_1 and R_2 , such that the new itinerary then first visits R_1 , then the state N (which is one new region) and then R_2 .

In total, we have two more regions, so the bigger country with $n + 1$ states is divided into $2n - 1 + 2 = 2(n + 1) - 1$ regions.

What also needs to be proved: Every connected country can be constructed by adding one state at a time, and that the division of R into R_1 and R_2 can always be done.

11. MISCELLANEOUS

Problem. 49

On a blackboard, there are some positive integers written. You may erase any integer k , and instead write any list of positive integers which are all strictly less than the erased number k . Of course, if you erase a 1, you are not allowed to write any number to the board.

Show that eventually all numbers will be erased.

Solution. 49

We first note that the set of possible blackboards is countable. We can represent a blackboard by writing down all integers that appear, from largest to smallest with repetition. For example, a blackboard containing three 5's, two 2's and three 1's would be 55522111. We can then list all blackboards alphabetically:

$$\begin{array}{l} 1 \quad 1 \\ 2 \quad 11 \\ 3 \quad 111 \\ \vdots \\ \quad 2 \\ \quad 21 \\ \quad 211 \\ \vdots \end{array}$$

Compare blackboards by saying that $B' < B$ if B' appears above B in this alphabetical list. We are now ready to show that the erasing-game always end.

Base case: The first blackboard, 1. The only thing we can do is to erase that number and the game ends.

Strong induction hypothesis: Suppose the game ends for all blackboards before B in the list. We want to show that the game also ends if we start at B .

However, by erasing an occurrence of an integer in B and adding some smaller integers to the board can only result in a lexicographically smaller blackboard—make sure you understand why! For those blackboards, the induction hypothesis tells us that the game ends. Thus, the game must end starting from B .

The type of induction used here is called *transfinite induction*.